How Map Projections Transform Velocity Vectors

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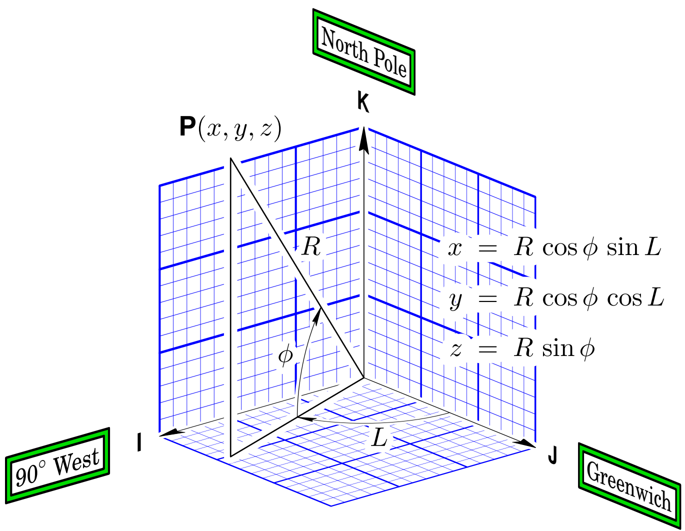
Let $\mathbf{P}$ be a point moving on the surface of the Earth, which we'll take to be a sphere of radius R. Suppose we're using a map projection given by two functions f and g like so:

$$x = f(\phi, L) \qquad \text{and} \qquad y = g(\phi, L).$$

Here (x, y) are (cartesian) coordinates on the map, and (ϕ, L) are, respectively, latitude and longitude on the Earth. (*Note:* As indicated in the figure below, we adopt the conventions that north latitude and west longitude are positive.) We want to know how the velocity vector of $\mathbf{P}$, as it moves on the Earth, is related to the velocity vector of its image point moving on the map. In other words, how does the map projection transform velocity vectors?

Velocity components of an object moving over the Earth are usually given as a component v_e in the (local) east direction and a component v_n in the north direction. These two values, along with the motion of $\mathbf{P}$ as a function of time and the map projection, are the given data for this problem. The first thing we must do is relate these given velocity components (v_e, v_n) to the time rates of change of latitude and longitude.



From the figure above, we have $\mathbf{P} = R \begin{bmatrix} \cos\phi \, \sin L \\ \cos\phi \, \cos L \\ \sin\phi \end{bmatrix}$. Differentiating this with respect to ϕ gives $\dfrac{\partial \mathbf{P}}{\partial \phi} = R \begin{bmatrix} -\sin\phi \, \sin L \\ -\sin\phi \, \cos L \\ \cos\phi \end{bmatrix}$, which has length R, so we can take $\mathbf{N} = \dfrac{1}{R} \dfrac{\partial \mathbf{P}}{\partial \phi}$ to be the definition of $\mathbf{N}$, the unit northward-pointing vector at $\mathbf{P}$'s location. Similarly, $\dfrac{\partial \mathbf{P}}{\partial L} = R \begin{bmatrix} \cos\phi \, \cos L \\ -\cos\phi \, \sin L \\ 0 \end{bmatrix}$ has length $R \cos\phi$ and points westward, so we can take $\mathbf{E} = -\dfrac{1}{R\cos\phi} \dfrac{\partial \mathbf{P}}{\partial L}$ to be the unit eastward pointing vector at $\mathbf{P}$. Notice that we have $\mathbf{E} \cdot \mathbf{N} = 0$, so the vectors $\mathbf{E}$ and $\mathbf{N}$ are orthogonal, as they should be.

Using these results we can relate the velocity vector $\dot{\mathbf{P}} = \dfrac{d\mathbf{P}}{dt}$ to $\dot{\phi}$ and $\dot{L}$ as follows:

$$\begin{aligned}\dot{\mathbf{P}} &= \frac{\partial \mathbf{P}}{\partial \phi}\,\dot{\phi} + \frac{\partial \mathbf{P}}{\partial L}\,\dot{L} \\ &= (R\,\mathbf{N})\,\dot{\phi} + (-R\cos\phi\,\mathbf{E})\,\dot{L} \\ &= \left(-R\cos\phi\,\dot{L}\right)\mathbf{E} + \left(R\,\dot{\phi}\right)\mathbf{N}\end{aligned}$$

But from the definition of v_e and v_n we know that $\dot{\mathbf{P}} = v_e\,\mathbf{E} + v_n\,\mathbf{N}$. Thus we must have $v_e = -R\cos\phi\,\dot{L}$ and $v_n = R\,\dot{\phi}$.

The rest is straightforward. Differentiating $x = f(\phi, L)$ using the chain rule, we have

$$\begin{aligned}\dot{x} &= \frac{\partial f}{\partial \phi}\,\dot{\phi} + \frac{\partial f}{\partial L}\,\dot{L} \\ &= \frac{\partial f}{\partial \phi}\left(\frac{v_n}{R}\right) + \frac{\partial f}{\partial L}\left(-\frac{v_e}{R\cos\phi}\right) \\ &= \left(-\frac{1}{R\cos\phi}\,\frac{\partial f}{\partial L}\right)v_e + \left(\frac{1}{R}\,\frac{\partial f}{\partial \phi}\right)v_n\end{aligned}$$

where we have used the above expressions for v_e and v_n to eliminate $\dot{\phi}$ and $\dot{L}$. A similar result holds for $\dot{y}$ and so we have the relationship between (v_e, v_n) and $(\dot{x}, \dot{y})$ given by the following matrix equation:

$$\boxed{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-\dfrac{1}{R\cos\phi}\,\dfrac{\partial f}{\partial L} & \dfrac{1}{R}\,\dfrac{\partial f}{\partial \phi} \\[2ex] -\dfrac{1}{R\cos\phi}\,\dfrac{\partial g}{\partial L} & \dfrac{1}{R}\,\dfrac{\partial g}{\partial \phi}\end{bmatrix}\begin{bmatrix}v_e\\ v_n\end{bmatrix}}$$

This is the result we were after. Note that the 2×2 matrix above (we'll call it $\mathbf{M}$) is *not* simply the Jacobian matrix of the map projection. This is because $\mathbf{E}$ and $\mathbf{N}$ are not the (ϕ, L)-coordinate frame on the Earth. The above matrix is the Jacobian matrix composed (on the right) with a change-of-basis matrix. The velocity transformation is really just the derivative of the map projection, although since we're not using a coordinate frame on the Earth, we get a matrix different from the Jacobian matrix representing it.

As an aside, here's something to think about: the vectors $\mathbf{E}$ and $\mathbf{N}$ on the Earth do not constitute the (ϕ, L)-coordinate frame, but maybe they're the coordinate frame field for some other set of coordinates? The answer is "No," but how do we know that? *Hint:* I'm not thinking of curvature here.

We'll spend the rest of this writeup doing examples illustrating the calculation of the matrix $\mathbf{M}$ for several projections in common use: Lambert, Stereographic, Mercator, Plate Carée and Orthographic. We'll also do a couple of plotting examples.

➢ **Example 1 :** Lambert Conformal Projection. This will actually be the hardest example of all, because the calculations here are more difficult than for our other examples. Ignoring, for simplicity, overall translations, rotations and scale factors, we have $x = r\sin\beta$ and $y = r\cos\beta$, where

$$r = \left[\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\right]^c \qquad \text{and} \qquad \beta = c\,L$$

where c is the cone constant. For our purposes, the key property of $r(\phi)$ here is that $r'(\phi) = \left(-\dfrac{c}{\cos\phi}\right) r(\phi)$. Calculating derivatives, we get:

$$\frac{\partial f}{\partial \phi} = \frac{\partial}{\partial \phi}(r\sin\beta) = \frac{dr}{d\phi}\sin\beta = \left(-\frac{c}{\cos\phi}\right) r\sin\beta$$

$$\frac{\partial f}{\partial L} = \frac{\partial}{\partial L}(r\sin\beta) = r\,\frac{d}{dL}(\sin\beta) = r\cos\beta\frac{d\beta}{dL} = c\,r\cos\beta$$

$$\frac{\partial g}{\partial \phi} = \frac{\partial}{\partial \phi}(r\cos\beta) = \frac{dr}{d\phi}\cos\beta = \left(-\frac{c}{\cos\phi}\right) r\cos\beta$$

$$\frac{\partial g}{\partial L} = \frac{\partial}{\partial L}(r\cos\beta) = r\,\frac{d}{dL}(\cos\beta) = -r\sin\beta\,\frac{d\beta}{dL} = -c\,r\sin\beta$$

and so our matrix **M** becomes

$$\begin{bmatrix} -\dfrac{1}{R\,\cos\phi}\dfrac{\partial f}{\partial L} & \dfrac{1}{R}\dfrac{\partial f}{\partial \phi} \\[2ex] -\dfrac{1}{R\,\cos\phi}\dfrac{\partial g}{\partial L} & \dfrac{1}{R}\dfrac{\partial g}{\partial \phi} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R\,\cos\phi}\,(c\,r\cos\beta) & \dfrac{1}{R}\left(-\dfrac{c}{\cos\phi}\right) r\sin\beta \\[2ex] -\dfrac{1}{R\,\cos\phi}\,(-c\,r\sin\beta) & \dfrac{1}{R}\left(-\dfrac{c}{\cos\phi}\right) r\cos\beta \end{bmatrix}$$

$$= \frac{c\,r}{R\,\cos\phi}\begin{bmatrix} -\cos\beta & -\sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}$$

$$= \frac{c\,r}{R\,\cos\phi}\begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

where $\gamma = \beta + 180°$. So the end result is a rotation through an angle of γ (which is a function of L only), and a scaling by $\dfrac{c\,r}{R\,\cos\phi}$, (which is a function of ϕ only). Note that we can only express **M** as a combination of a rotation and a scaling because the Lambert projection is conformal. It's only for conformal projections that the scale factor at each point is the same in all directions at that point.

➢ **Example 2 :** Polar Stereographic Projection. Again ignoring overall translations, rotations and scale factors, we have $x = r\sin\beta$ and $y = r\cos\beta$, where

$$r = \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \qquad \text{and} \qquad \beta = L$$

Note that this is the limiting form of the Lambert equations when $c \to 1$. So here we can just use the results for the Lambert Conformal Projection example, replacing c by 1 everywhere.

➢ **Example 3 :** Mercator Projection. As usual we'll ignore overall translations, rotations and scale factors. Then $x = -L$ and $y = \log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$. Note that $\frac{dy}{d\phi} = \frac{1}{\cos\phi}$. Calculating the partial derivatives of f and g we get:

$$\frac{\partial f}{\partial \phi} = \frac{\partial}{\partial \phi}(-L) = 0$$

$$\frac{\partial f}{\partial L} = \frac{\partial}{\partial L}(-L) = -1$$

$$\frac{\partial g}{\partial \phi} = \frac{dy}{d\phi} = \frac{1}{\cos\phi}$$

$$\frac{\partial g}{\partial L} = \frac{\partial}{\partial L}\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = 0$$

and so our matrix **M** is

$$\begin{bmatrix} -\frac{1}{R\,\cos\phi}\frac{\partial f}{\partial L} & \frac{1}{R}\frac{\partial f}{\partial \phi} \\ -\frac{1}{R\,\cos\phi}\frac{\partial g}{\partial L} & \frac{1}{R}\frac{\partial g}{\partial \phi} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R\,\cos\phi}(-1) & \frac{1}{R}(0) \\ -\frac{1}{R\,\cos\phi}(0) & \frac{1}{R}\left(\frac{1}{\cos\phi}\right) \end{bmatrix}$$

$$= \frac{1}{R\,\cos\phi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The end result is a scaling by $(R\,\cos\phi)^{-1}$ and no rotation. (Or, if you like, a rotation by an angle of zero.)

➢ **Example 4 :** Plate Carée Projection. Once again we'll ignore overall translations, rotations and scale factors. Then $x = -L$ and $y = \phi$. Turning the crank, we get:

$$\begin{aligned}
\frac{\partial f}{\partial \phi} &= \frac{\partial}{\partial \phi}(-L) = 0 \\
\frac{\partial f}{\partial L} &= \frac{\partial}{\partial L}(-L) = -1 \\
\frac{\partial g}{\partial \phi} &= \frac{\partial}{\partial \phi}(\phi) = 1 \\
\frac{\partial g}{\partial L} &= \frac{\partial}{\partial L}(\phi) = 0
\end{aligned}$$

and so our matrix $\mathbf{M}$ is

$$\begin{aligned}
\begin{bmatrix} -\dfrac{1}{R\cos\phi}\dfrac{\partial f}{\partial L} & \dfrac{1}{R}\dfrac{\partial f}{\partial \phi} \\ -\dfrac{1}{R\cos\phi}\dfrac{\partial g}{\partial L} & \dfrac{1}{R}\dfrac{\partial g}{\partial \phi} \end{bmatrix} &= \begin{bmatrix} -\dfrac{1}{R\cos\phi}(-1) & \dfrac{1}{R}(0) \\ -\dfrac{1}{R\cos\phi}(0) & \dfrac{1}{R}(1) \end{bmatrix} \\
&= \frac{1}{R}\begin{bmatrix} \sec\phi & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

It's not hard to show that this matrix *cannot* be expressed as the composition of a rotation and a scaling. That's because the Plate Carée projection is not conformal.

➢ **Example 5 :** Orthographic Projection. As before, we'll ignore overall translations, rotations and scale factors. Then $x = \mathbf{P}\cdot\mathbf{A}$ and $y = \mathbf{P}\cdot\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are two orthonormal vectors in $\mathbb{R}^3$. Thus we have

$$\begin{aligned}
\frac{\partial f}{\partial \phi} &= \frac{\partial}{\partial \phi}(\mathbf{P}\cdot\mathbf{A}) = \left(\frac{\partial \mathbf{P}}{\partial \phi}\right)\cdot\mathbf{A} = R\,\mathbf{N}\cdot\mathbf{A} \\
\frac{\partial f}{\partial L} &= \frac{\partial}{\partial L}(\mathbf{P}\cdot\mathbf{A}) = \left(\frac{\partial \mathbf{P}}{\partial L}\right)\cdot\mathbf{A} = -R\cos\phi\,\mathbf{E}\cdot\mathbf{A} \\
\frac{\partial g}{\partial \phi} &= \frac{\partial}{\partial \phi}(\mathbf{P}\cdot\mathbf{B}) = \left(\frac{\partial \mathbf{P}}{\partial \phi}\right)\cdot\mathbf{B} = R\,\mathbf{N}\cdot\mathbf{B} \\
\frac{\partial g}{\partial L} &= \frac{\partial}{\partial L}(\mathbf{P}\cdot\mathbf{B}) = \left(\frac{\partial \mathbf{P}}{\partial L}\right)\cdot\mathbf{B} = -R\cos\phi\,\mathbf{E}\cdot\mathbf{B}
\end{aligned}$$

and so our matrix is

$$\begin{bmatrix} -\dfrac{1}{R\,\cos\phi}\,\dfrac{\partial f}{\partial L} & \dfrac{1}{R}\,\dfrac{\partial f}{\partial \phi} \\[2ex] -\dfrac{1}{R\,\cos\phi}\,\dfrac{\partial g}{\partial L} & \dfrac{1}{R}\,\dfrac{\partial g}{\partial \phi} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R\,\cos\phi}\,(-R\cos\phi\,\mathbf{E}\cdot\mathbf{A}) & \dfrac{1}{R}\,(R\,\mathbf{N}\cdot\mathbf{A}) \\[2ex] -\dfrac{1}{R\,\cos\phi}\,(-R\cos\phi\,\mathbf{E}\cdot\mathbf{B}) & \dfrac{1}{R}\,(R\,\mathbf{N}\cdot\mathbf{B}) \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{E}\cdot\mathbf{A} & \mathbf{N}\cdot\mathbf{A} \\ \mathbf{E}\cdot\mathbf{B} & \mathbf{N}\cdot\mathbf{B} \end{bmatrix}$$

which worked out pretty nicely. It's interesting to note that the determinant of this matrix is

$$\begin{aligned}(\mathbf{E}\cdot\mathbf{A})\,(\mathbf{N}\cdot\mathbf{B}) - (\mathbf{E}\cdot\mathbf{B})\,(\mathbf{N}\cdot\mathbf{A}) &= (\mathbf{E}\times\mathbf{N})\cdot(\mathbf{A}\times\mathbf{B}) \\ &= \mathbf{P}\cdot(\mathbf{A}\times\mathbf{B})\end{aligned}$$

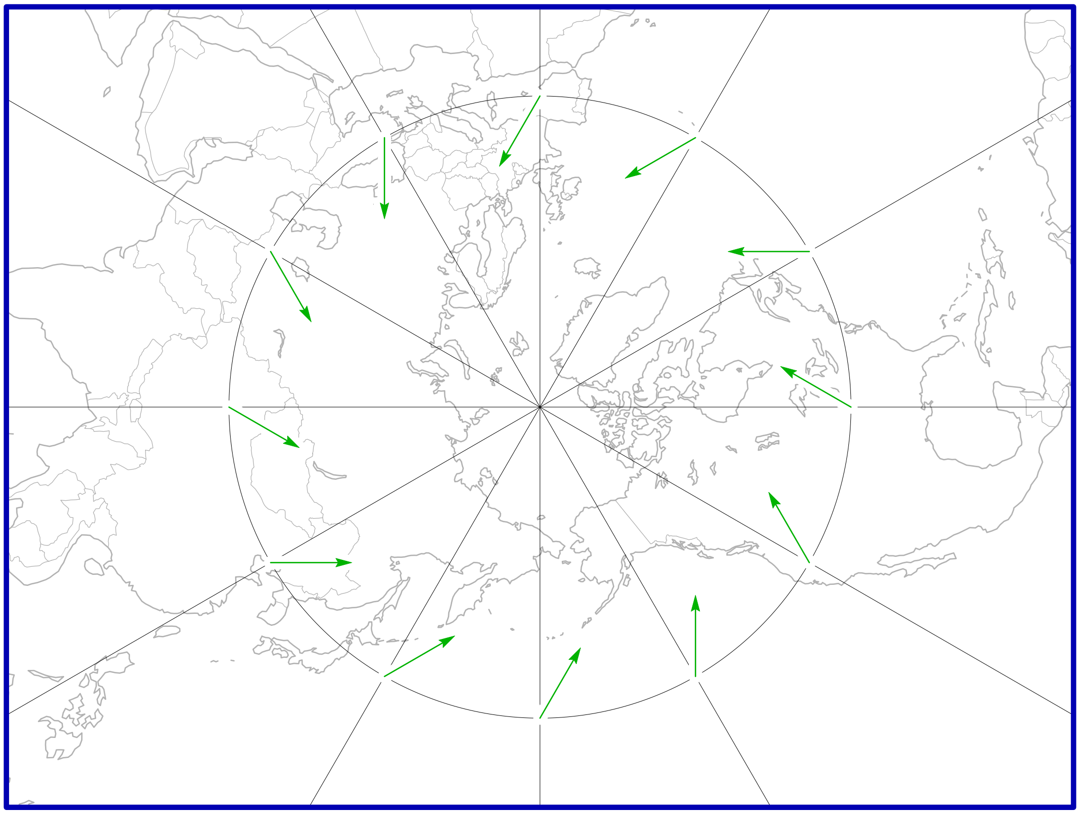
where we have set $R = 1$ for convenience (otherwise we would have $\mathbf{E}\times\mathbf{N} = \mathbf{P}/R$). Now, $\mathbf{A}$ and $\mathbf{B}$ span the picture plane, so this says the matrix becomes singular when $\mathbf{P}$ lies in that plane, or in other words, for points $\mathbf{P}$ on the "rim" of the Earth, as seen in this projection. This is not surprising. The Orthographic Projection ceases to be a parametrization of the Earth's surface there, so it's derivative (the velocity transformation) can't be expected to be well-behaved at such points.

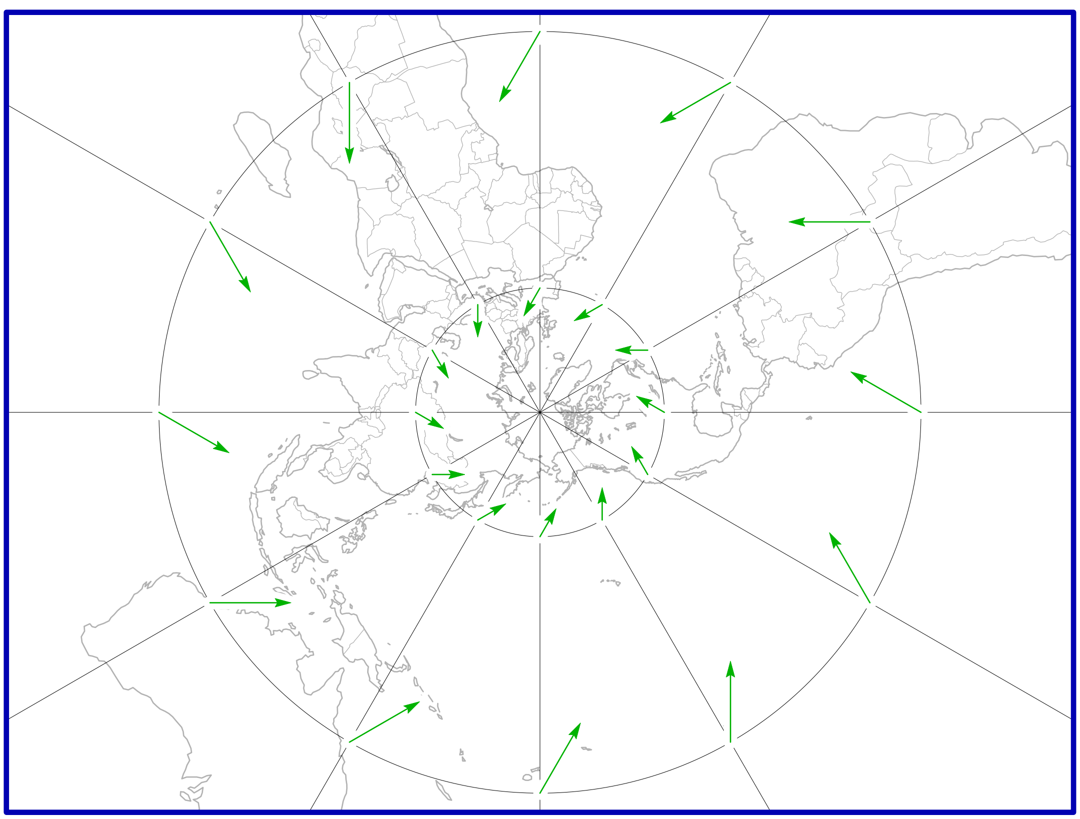
➢ **Example 6 :** Enough equations. Let's draw some pictures. We'll use a Polar Stereographic projection, and we'll choose a tangent vector field on the Earth that has a constant bearing of $30°$, and also a constant magnitude.

The top plot on the next page shows the vectors drawn on the $40°$ latitude circle, at multiples of $30°$ longitude. Notice that the velocity transformation takes care of the direction of the image vector on the map.

The bottom plot is drawn to a different overall scale. The vectors are drawn on a $40°$ latitude circle and also a $-20°$ latitude circle. The farther out from the center of the map we get, the larger the *local* map scale is, so if a vector has a fixed *true* length, it should be drawn longer where the map scale is larger. Here, the ratio of the scale factors at the two latitudes is about 2.5. The velocity transformation handles this rescaling automatically.

Note again that we can only break up the velocity transformation into a rotation followed by a scaling because the Polar Stereographic projection is conformal. If we were using a non-conformal projection, the velocity transformation would still handle the length and direction of the image vector correctly, but you couldn't view the transformation as simply a rotation and a scaling.





➢ **Example 7 :** Another plotting example, this time with the Orthographic Projection. We chose a point on the Earth with latitude $40°$ and longitude $105°$. This point will be in the center of the map. We then took **A** to be the east vector **E** for that location, and then **B** to be the north vector **N** there. As in the previous example, we took the vector field to have a constant bearing of $30°$ and a constant magnitude.

The Orthographic Projection is not conformal, so we can't talk about rotation angles and scale factors at any point, but the velocity transformation still correctly gives the direction and size of the plotted vector.