Moments, Centroids and Axis Angles

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This write-up concerns the calculation of moments, centroids and axes for objects defined either as a collection of grid squares, or as closed polylines. Actually, only the calculation of moments differs for the two cases—once the moments are in hand, calculation of centroids and axes proceeds in the same way for both situations.

Moments for Objects defined as Unions of Grid Squares

Suppose we have a two-dimensional grid $G = \Big\{ (x, y) \in \mathbb{Z}^2 \mid 0 \le x < N_x,\ 0 \le y < N_y \Big\}$ and a region of interest (object) $B \subset G$. We can define a function $f : G \to \{0, 1\} \subset \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in B \\ 0 & \text{if } (x,y) \notin B \end{cases}$$

(Note: statisticians would call f the "indicator function" of B, while mathematicians would call f the "characteristic function" of B.) We can then define moments of B by summing certain functions over the grid.

First the area of the object. This will be expressed simply as a count of the number of grid squares occupied by the object, and can be thought of as a kind of "zero-th moment" of the object.

$$A = \sum_{x,y} f(x,y)$$

There are two first moments, one in x and one in y, denoted respectively by S_x and S_y, defined by

$$S_x = \sum_{x,y} x\, f(x,y) \qquad \text{and} \qquad S_y = \sum_{x,y} y\, f(x,y)$$

There are four 2^{nd} moments (although only three are in general distinct) defined by

$$S_{xx} = \sum_{x,y} x^2\, f(x,y) \qquad S_{xy} = S_{yx} = \sum_{x,y} xy\, f(x,y) \qquad S_{yy} = \sum_{x,y} y^2\, f(x,y)$$

We will not investigate moments higher than the 2^{nd} in this write-up.

Moments for Objects defined by Closed Polylines

Now we consider our object B to be the region inside a closed polyline (or perhaps the union of several such). Surprisingly, the situation here (at least from a computational point of view) is really no more complicated than it was for objects on a grid. Deriving the requisite formulas is more involved, but the actual computations really only involve looping around the vertices of the polyline and performing some simple arithmetic on the x and y coordinates of the vertices.

Let's start, as before, by looking at area. The area form on the plane $\mathbb{R}^2$ is $\Omega = dx \wedge dy$. We can use Stokes' Theorem to turn the integral of Ω over B into an integral around the enclosing polyline of another form α such that $d\alpha = \Omega$. Of course, we need to know that Stokes' Theorem holds for regions whose boundaries are only piecewise smooth, but fortunately it does.

There are many choices for α. We will use $\alpha = (1/2)\,(x\,dy - y\,dx)$. It is now necessary to integrate this form around the closed polyline bounding our object. We do this by integrating α over each of the edges of the polyline, and then summing the results.

Let's say the polyline has n vertices (x_i, y_i) with $i \in \mathbb{Z}/n\mathbb{Z}$. If we consider edge $\#i$ as being the straight line segment running from (x_i, y_i) to (x_{i+1}, y_{i+1}), we can then parametrize this segment as follows:

$$\{\,(1-t)\,(x_i, y_i) + t\,(x_{i+1}, y_{i+1}) \mid 0 \le t \le 1\}$$

Using this parametrization we can pull α back to a form involving t, which (after some cancellation) simplifies down to $(1/2)\,(\,x_i\,y_{i+1} - y_i\,x_{i+1}\,)\,dt$. Finally, we integrate this over $0 \le t \le 1$ to get $(1/2)\,(\,x_i\,y_{i+1} - y_i\,x_{i+1}\,)$. The area of the object is now gotten by summing over the edges:

$$A = \frac{1}{2} \sum_i (x_i\,y_{i+1} - y_i\,x_{i+1})$$

A few points to note here: first, our integral over the bounding polyline is an *oriented* integral. If the direction (clockwise *vs.* counter-clockwise) in which the polyline is traversed is changed, the integral will change sign. Therefore to get the true geometrical area we should take absolute values after the summation. Second, the summation index i takes values in $\mathbb{Z}/n\mathbb{Z}$ rather than $\mathbb{Z}$. Thus if $i = n - 1$, then $i + 1$ is to be interpreted as 0, not n.

We can do similar things for higher moments. We will simply present the results here, rather than plodding through each derivation separately. In the table below, we indicate the form β which would be integrated over the object to define the moment, the form α, chosen so that $d\alpha = \beta$, integrated around the bounding polyline (Stokes' Theorem again), and the i^{th} term in the resulting summation.

β	α	*Summation Term*
$dx \wedge dy$	$\frac{(\,x\,dy - y\,dx\,)}{2}$	$\frac{(\,x_i\,y_{i+1} - y_i\,x_{i+1}\,)}{2}$
$x\,dx \wedge dy$	$\frac{x^2\,dy}{2}$	$\frac{\left(x_i^2 + x_i\,x_{i+1} + x_{i+1}^2\right)\,(y_{i+1} - y_i)}{6}$
$y\,dx \wedge dy$	$-\frac{y^2\,dx}{2}$	$\frac{(\,x_i - x_{i+1}\,)\,\left(y_i^2 + y_i\,y_{i+1} + y_{i+1}^2\right)}{6}$
$x^2\,dx \wedge dy$	$\frac{x^3\,dy}{3}$	$\frac{\left(x_i^3 + x_i^2\,x_{i+1} + x_i\,x_{i+1}^2 + x_{i+1}^3\right)\,(y_{i+1} - y_i)}{12}$
$y^2\,dx \wedge dy$	$-\frac{y^3\,dx}{3}$	$\frac{(\,x_i - x_{i+1}\,)\,\left(y_i^3 + y_i^2\,y_{i+1} + y_i\,y_{i+1}^2 + y_{i+1}^3\right)}{12}$
$xy\,dx \wedge dy$	$\frac{xy\,(\,x\,dy - y\,dx\,)}{4}$	$\frac{(x_i\,y_{i+1} - y_i\,x_{i+1}\,)\,(\,2\,x_i\,y_i + x_{i+1}\,y_i + x_i\,y_{i+1} + 2\,x_{i+1}\,y_{i+1}\,)}{24}$

Looking at this table, the reader can easily see why we are not considering moments higher than the 2^{nd} in this write-up . . . there would hardly be room to write them down.

Now that we know how to calculate moments for both classes of objects—those that are given on a grid and those given as polyline boundaries, it's worthwhile to consider moments more generally. Up until now, we have been working with moments *about the origin*. What about moments about another point? And what if our object moves or rotates? Do we have to calculate the moments all over again from scratch? It would be nice if there were some way to calculates new moments from old. This can be done provided we restrict ourselves a simple enough class of coordinate transformations. In this write-up we will work with *affine* transformations.

Transformation of Moments

Suppose we have a coordinate transformation $(x, y) \mapsto (u, v)$ defined by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

where the m's and b's are given constants. This is called an affine coordinate transformation. Translations, rotations, reflections, scaling and shear are all special cases of this general form. How are the u, v moments related to the x, y moments? Let ϕ denote the transformation, so that $\phi(x, y) = (u, v)$. It all really boils down to examining pullbacks under ϕ.

Let $M = [m_{ij}]$ be the 2×2 coefficient matrix above, and let $\Delta = \det M$. Then we have, *e.g.*,

$$\begin{aligned} \phi^* (du \wedge dv) &= (m_{11}\, dx + m_{12}\, dy) \wedge (m_{21}\, dx + m_{22}\, dy) \\ &= (m_{11}\, m_{22} - m_{21}\, m_{12})\, dx \wedge dy \\ &= \Delta\, dx \wedge dy \end{aligned}$$

If we let A' be the (signed) area of the image object $\phi(B)$, then we have from the change of variables formula for integrals that

$$A' = \int_{\phi(B)} du \wedge dv = \int_B \phi^* (du \wedge dv) = \Delta \int_B dx \wedge dy = \Delta\, A$$

What about first moments? We have $\phi^* (u\, du \wedge dv) = (m_{11}\, x + m_{12}\, y + b_x)\ \Delta\, dx \wedge dy$, and so integrating as above we get $S_u = \Delta\ (m_{11}\, S_x + m_{12}\, S_y + b_x\, A)$. Similarly, $S_v = \Delta\ (m_{21}\, S_x + m_{22}\, S_y + b_y\, A)$.

Second moments are messier. We have $\phi^* (u^2\, du \wedge dv) = (m_{11}\, x + m_{12}\, y + b_x)^2\ \Delta\, dx \wedge dy$, from which

$$S_{uu} = \Delta\ \left(m_{11}^2\, S_{xx} + m_{12}^2\, S_{yy} + b_x^2\, A + 2\, m_{11}\, m_{12}\, S_{xy} + 2\, m_{11}\, b_x\, S_x + 2\, m_{12}\, b_x\, S_y\right)$$

Similarly,

$$S_{vv} = \Delta\ \left(m_{21}^2\, S_{xx} + m_{22}^2\, S_{yy} + b_y^2\, A + 2\, m_{21}\, m_{22}\, S_{xy} + 2\, m_{21}\, b_y\, S_x + 2\, m_{22}\, b_y\, S_y\right)$$

and

$$\begin{aligned} S_{uv} = \Delta\ (&m_{11}\, m_{21}\, S_{xx} + m_{12}\, m_{22}\, S_{yy} + (m_{11}\, m_{22} + m_{12}\, m_{21})\ S_{xy} \\ &+ (m_{21}\, b_x + m_{11}\, b_y)\ S_x + (m_{22}\, b_x + m_{12}\, b_y)\ Sy + b_x\, b_y\, A\,) \end{aligned}$$

Note that for many special transformations, like translations and rotations, $\Delta = 1$.

Centroid and Axis

It is possible to translate coordinates so that $S_u = S_v = 0$. When this condition holds, the origin of the coordinate system is at the *centroid* of the object. To find the coordinates $(\overline{x}, \overline{y})$ of the centroid, we transform by

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} b_x \\ b_y \end{bmatrix} = -\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}$$

Then $\Delta = 1$ and we have $S_u = S_x - \overline{x}\,A$ and $S_v = S_y - \overline{y}\,A$. The condition that S_u and S_v both be zero now gives

$$(\overline{x}, \overline{y}) = \left(\frac{S_x}{A}, \frac{S_y}{A}\right)$$

Now let's consider the axis. To do this, we will suppose that the origin our coordinate system is at the object's centroid. This means that $S_x = S_y = 0$. We'll also confine ourselves to rotations of the coordinate system, so that $M = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for some angle θ, and $b_x = b_y = 0$. Then $\Delta = 1$ and the transformation laws for 2^{nd} moments become

$$\begin{aligned} S_{uu} &= \cos^2\theta\, S_{xx} + \sin^2\theta\, S_{yy} + 2\,\cos\theta\,\sin\theta\, S_{xy} \\ S_{vv} &= \sin^2\theta\, S_{xx} + \cos^2\theta\, S_{yy} - 2\,\cos\theta\,\sin\theta\, S_{xy} \\ S_{uv} &= -\cos\theta\,\sin\theta\, S_{xx} + \cos\theta\,\sin\theta\, S_{yy} + \left(\cos^2\theta - \sin^2\theta\right)\, S_{xy} \end{aligned}$$

The urge to jump right in with trig identities and simplify this is strong, but we will resist for the moment, for a better simplification is available. Let $\sigma_0 = S_{uu} + S_{vv}$, and $\rho_0 = S_{xx} + S_{yy}$. Further, let $\sigma_1 = (S_{uu} - S_{vv})\,/2$, $\rho_1 = (S_{xx} - S_{yy})\,/2$ and $\sigma_2 = S_{uv}$, $\rho_2 = S_{xy}$. Rewriting the above transformation laws in terms of the σ's and ρ's (and *now* using trig identities) we get a great improvement:

$$\sigma_0 = \rho_0$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

We'll define the object axis by requiring that S_{uu} (considered as a function of θ) be maximized. Taking the above expression for S_{uu} and differentiating, we have, as the reader can verify, $S'_{uu} = 2\,\sigma_2$, and $S''_{uu} = -4\,\sigma_1$. By elementary calculus, for S_{uu} to be a maximum we must have $S'_{uu} = 0$ and $S''_{uu} < 0$. We therefore determine the axis angle θ by two conditions—first, that σ_2 be zero, and second, that σ_1 be positive.

The above matrix equation is easy to invert:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$$

and upon setting $\sigma_2 = 0$ we have that $\rho_1 = \sigma_1\,\cos 2\theta$ and $\rho_2 = \sigma_1\,\sin 2\theta$. It's tempting to simply conclude that $\tan 2\theta = \rho_2/\rho_1$, but what about the quadrant of θ? Since $\sigma_1 > 0$, we can write

$$\theta = \frac{1}{2}\,\arg\left(\rho_1 + \rho_2\, i\right)$$

Many programming languages have a 2-argument arctangent function, usually called `atan2`. The expression `atan2(y,x)` resolves the quadrant of the point (x, y) correctly (though the fact that the order of x and y is reversed in the argument list is a continual nuisance). Thus we finally have

$$\theta = \texttt{0.5*atan2(}\rho_2\texttt{,}\rho_1\texttt{)}$$

and this is our axis angle.