Least-Squares Circle Fit

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Given a finite set of points in $\mathbb{R}^2$, say $\{\,(x_i, y_i)\,|\, 0 \le i < N\,\}$, we want to find the circle that "best" (in a least-squares sense) fits the points. Define

$$\overline{x} = \frac{1}{N}\sum_i x_i \quad \text{and} \quad \overline{y} = \frac{1}{N}\sum_i y_i$$

and let $u_i = x_i - \overline{x}, \quad v_i = y_i - \overline{y}$ for $0 \le i < N$. We solve the problem first in (u, v) coordinates, and then transform back to (x, y).

Let the circle have center (u_c, v_c) and radius R. We want to minimize $S = \sum_i \left(g(u_i, v_i) \right)^2$, where $g(u, v) = (u - u_c)^2 + (v - v_c)^2 - \alpha$, and where $\alpha = R^2$. To do that, we differentiate $S(\alpha, u_c, v_c)$.

$$\begin{aligned} \frac{\partial S}{\partial \alpha} &= 2\sum_i g(u_i, v_i)\,\frac{\partial g}{\partial \alpha}(u_i, v_i) \\ &= -2\sum_i g(u_i, v_i) \end{aligned}$$

Thus $\partial S/\partial\alpha = 0$ iff

$$\boxed{\sum_i g(u_i, v_i) = 0} \qquad \textbf{Eq. 1}$$

Continuing, we have

$$\begin{aligned} \frac{\partial S}{\partial u_c} &= 2\sum_i g(u_i, v_i)\,\frac{\partial g}{\partial u_c}(u_i, v_i) \\ &= 2\sum_i g(u_i, v_i)\,2(u_i - u_c)(-1) \\ &= -4\sum_i (u_i - u_c)\,g(u_i, v_i) \\ &= -4\sum_i u_i\,g(u_i, v_i) + 4\,u_c \underbrace{\sum_i g(u_i, v_i)}_{=\ 0 \text{ by } \textbf{Eq. 1}} \end{aligned}$$

Thus, in the presence of **Eq. 1**, $\partial S/\partial u_c = 0$ holds iff

$$\boxed{\sum_i u_i\,g(u_i, v_i) = 0} \qquad \textbf{Eq. 2}$$

Similarly, requiring $\partial S/\partial v_c = 0$ gives

$$\boxed{\sum_i v_i\,g(u_i, v_i) = 0} \qquad \textbf{Eq. 3}$$

Expanding **Eq. 2** gives

$$\sum_i u_i \left[u_i^2 - 2\, u_i\, u_c + u_c^2 + v_i^2 - 2\, v_i\, v_c + v_c^2 - \alpha \right] = 0$$

Defining $S_u = \sum_i u_i, \ S_{uu} = \sum_i u_i^2,$ *etc.*, we can rewrite this as

$$S_{uuu} - 2\, u_c\, S_{uu} + u_c^2\, S_u + S_{uvv} - 2\, v_c\, S_{uv} + v_c^2\, S_u - \alpha\, S_u = 0$$

Since $S_u = 0$, this simplifies to

$$\boxed{u_c\, S_{uu} + v_c\, S_{uv} = \frac{1}{2}\left(S_{uuu} + S_{uvv}\right)} \qquad \textbf{Eq. 4}$$

In a similar fashion, expanding **Eq. 3** and using $S_v = 0$ gives

$$\boxed{u_c\, S_{uv} + v_c\, S_{vv} = \frac{1}{2}\left(S_{vvv} + S_{vuu}\right)} \qquad \textbf{Eq. 5}$$

Solving **Eq. 4** and **Eq. 5** simultaneously gives (u_c, v_c). Then the center (x_c, y_c) of the circle in the original coordinate system is $(x_c, y_c) = (u_c, v_c) + (\overline{x}, \overline{y})$.

To find the radius R, expand **Eq. 1**:

$$\sum_i \left[u_i^2 - 2\, u_i\, u_c + u_c^2 + v_i^2 - 2\, v_i\, v_c + v_c^2 - \alpha \right] = 0$$

Using $S_u = S_v = 0$ again, we get

$$N\, \left(u_c^2 + v_c^2 - \alpha \right) + S_{uu} + S_{vv} = 0$$

Thus

$$\boxed{\alpha = u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N}} \qquad \textbf{Eq. 6}$$

and, of course, $R = \sqrt{\alpha}$.

See the next page for an example!

Example : Let's take a few points from the parabola $y = x^2$ and fit a circle to them. Here's a table giving the points used:

i	x_i	y_i	u_i	v_i
0	0.000	0.000	-1.500	-3.250
1	0.500	0.250	-1.000	-3.000
2	1.000	1.000	-0.500	-2.250
3	1.500	2.250	0.000	-1.000
4	2.000	4.000	0.500	0.750
5	2.500	6.250	1.000	3.000
6	3.000	9.000	1.500	5.750

Here we have $N = 7$, $\overline{x} = 1.5$, and $\overline{y} = 3.25$. Also, $S_{uu} = 7$, $S_{uv} = 21$, $S_{vv} = 68.25$, $S_{uuu} = 0$, $S_{vvv} = 143.81$, $S_{uvv} = 31.5$, $S_{vuu} = 5.25$. Thus (using **Eq. 4** and **Eq. 5**) we have the following 2×2 linear system for (u_c, v_c):

$$\begin{bmatrix} 7 & 21 \\ 21 & 68.25 \end{bmatrix} \begin{bmatrix} u_c \\ v_c \end{bmatrix} = \begin{bmatrix} 15.75 \\ 74.531 \end{bmatrix}$$

Solving this system gives $(u_c, v_c) = (-13.339, 5.1964)$, and thus $(x_c, y_c) = (-11.839, 8.4464)$. Substituting these values into **Eq. 6** gives $\alpha = 215.69$, and hence $R = 14.686$. A plot of this example appears below.